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Note

A note on distance matrices yielding elementary landscapes for the TSP

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Symmetric and antisymmetric distance matrices in the single agent traveling salesman problem (TSP) are not the only distance matrices to generate elementary landscapes for "swap" and "2-opt" neighborhoods.

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In a TSP with n cities, there are $(n-1)!$ possible tours and a tour's cost is determined from the $n \times n$ distance matrix $\mathbf{D}$ by

$$f_{\mathbf{D}}(\pi) = \sum_{i=1}^{n} D_{i,\pi(i)}.$$

The *landscape* determined by a neighborhood N and distance matrix $\mathbf{D}$ is the pair $(N, \mathbf{f_D})$, where $\mathbf{f_D}$ denotes the vector of tour costs.

Let L be the $(n-1)! \times (n-1)!$ Laplacian determined by N and let $\tilde{\mathbf{f}}_{\mathbf{D}} = \mathbf{f_D} - \boldsymbol{\mu}$ where $\boldsymbol{\mu}$ is a vector containing the mean value of $\mathbf{f_D}$ in each cell. A landscape is *elementary* if $\tilde{\mathbf{f}}$ is an eigenvector of L.

Limiting N to be either a swap or 2-opt neighborhood, Stadler [1] states that if $\mathbf{D}$ is a symmetric or antisymmetric distance matrix then $(N, \mathbf{f_D})$ is elementary. The symmetric case is a special case of the more general results presented in Colletti and Barnes [2].

Stadler [1] further claims the converse – if the landscape $(N, \mathbf{f_D})$ is elementary for swap or 2-opt neighborhoods, then the distance matrix $\mathbf{D}$ *must* be either symmetric or antisymmetric. As shown below, this latter claim is incorrect.

Define a *deformation pair* to be $\mathbf{q}, \mathbf{r} \in \mathbb{R}^n$ such that

$$\sum_{i=1}^{n} q_i + r_i = 0. \tag{1}$$

(Since setting r_n to be the negative of the sum of *any* choice of $q_1, \ldots, q_n, r_1, \ldots, r_{n-1}$ yields a $(\mathbf{q}, \mathbf{r})$ pair satisfying (1), infinitely many deformation pairs exist.)

Proposition 1. Let $(\mathbf{q}, \mathbf{r})$ be a deformation pair and define

$$\mathbf{Q} = \begin{pmatrix} q_1 & q_2 & \cdots & q_n \\ q_1 & q_2 & \cdots & q_n \\ \vdots & \vdots & \ddots & \vdots \\ q_1 & q_2 & \cdots & q_n \end{pmatrix} \quad \text{and} \quad \mathbf{R} = \begin{pmatrix} r_1 & r_1 & \cdots & r_1 \\ r_2 & r_2 & \cdots & r_2 \\ \vdots & \vdots & \ddots & \vdots \\ r_n & r_n & \cdots & r_n \end{pmatrix}.$$

If $\mathbf{D}$ yields an elementary landscape under neighborhood N, then so does $\mathbf{D} + \mathbf{Q} + \mathbf{R}$.

Proof. If $\mathbf{Q}, \mathbf{R}$ and $\mathbf{D}$ are as defined in the proposition, we only need to show that $\tilde{\mathbf{f}}_{\mathbf{D}} \equiv \tilde{\mathbf{f}}_{\mathbf{D}+\mathbf{Q}+\mathbf{R}}$. Since any $\mathbf{Q} + \mathbf{R}$ satisfying (1) generates a constant TSP, i.e., $\tilde{\mathbf{f}}_{\mathbf{Q}+\mathbf{R}} \equiv \mathbf{0}$ (see [3]), $\tilde{\mathbf{f}}_{\mathbf{D}} \equiv \tilde{\mathbf{f}}_{\mathbf{D}+\mathbf{Q}+\mathbf{R}}$. □

Theorem 2. Distance matrices yielding elementary landscapes exist which are neither symmetric nor antisymmetric.

Proof. Let $\mathbf{S}$ be any *symmetric* matrix with $S_{1,n-1} \neq -1/2$. Setting $r_1 = 1$, $q_n = -1$, $r_i = 0$ for $i > 1$ and $q_i = 0$ for $i < n$ yields a deformation pair, and asymmetric

$$\mathbf{R} + \mathbf{Q} = \begin{pmatrix} 1 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & -1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 \end{pmatrix}.$$

$\mathbf{T} = \mathbf{S} + \mathbf{R} + \mathbf{Q}$ is an asymmetric matrix yielding an elementary landscape. Showing that $\mathbf{T}$ is not antisymmetric completes the proof.

If $\mathbf{T}$ is antisymmetric, $T_{1,n-1} = -T_{n-1,1}$, which implies

$$S_{1,n-1} + 1 = -S_{n-1,1} = -S_{1,n-1}$$

so that

$$2S_{1,n-1} + 1 = 0,$$

contradicting the assumption that $S_{1,n-1} \neq -1/2$. Therefore, $\mathbf{T}$ is a distance matrix yielding an elementary landscape which is neither symmetric nor antisymmetric. □

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References

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